1. $X(u,v) = ((a+b\cos u)\cos v,\ (a+b\cos u)\sin v,\ b\sin u)$

$X_u = (-b\sin u\cos v,\ -b\sin u\sin v,\ b\cos u)$

$X_v = (-(a+b\cos u)\sin v,\ (a+b\cos u)\cos v,\ 0)$

$X_{uu} = (-b\cos u\cos v,\ -b\cos u\sin v,\ -b\sin v)$

$X_{vv} = (-(a+b\cos u)\cos v,\ -(a+b\cos v)\sin v,\ 0)$

$X_{uv} = (b\sin u\sin v,\ -b\sin u\cos v,\ 0)$

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{b(a+b\cos u)}\left(-b(a+b\cos u)\cos u\cos v,\ -b(a+b\cos u)\cos u\sin v,\ -b(a+b\cos u)\sin u\right)$$

$$= (-\cos u\cos v,\ -\cos u\sin v,\ \sin u)$$

$$h = \begin{pmatrix} \langle N, X_{uu}\rangle & \langle N, X_{uv}\rangle \\ \langle N, X_{uv}\rangle & \langle N, X_{vv}\rangle \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & (a+b\cos u)\cos u \end{pmatrix}$$

$$g = \begin{pmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \langle X_v, X_v\rangle \end{pmatrix} = \begin{pmatrix} b^2 & 0 \\ 0 & (a+b\cos u)^2 \end{pmatrix}$$

$$K = \frac{\det(h)}{\det(g)} = \frac{b(a+b\cos u)\cos u}{b^2(a+b\cos u)^2} = \frac{\cos u}{b(a+b\cos u)}$$

$$H = tr(g^{-1}h) = tr\left(\begin{pmatrix} \frac{1}{b^2} & 0 \\ 0 & \frac{1}{(a+b\cos u)^2} \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & (a+b\cos u)\cos u \end{pmatrix}\right)$$

$$= \frac{1}{b} + \frac{\cos u}{a+b\cos u}$$

If we take $N = -\frac{X_u \times X_v}{|X_u \times X_v|}$, then $H = -\frac{1}{b} - \frac{\cos u}{a+b\cos u}$, $K = \frac{\cos u}{b(a+b\cos u)}$

2a. Let $F(x,y,z) = x^2+y^2-z$, $S = F^{-1}(0)$

Choose $N = \frac{\nabla F}{|\nabla F|} = \frac{(2x, 2y, -1)}{\sqrt{4x^2+4y^2+1}}$

Take $x = r\cos\theta$, $y = r\sin\theta$

$N(r,\theta) = \frac{1}{\sqrt{4r^2+1}}(2r\cos\theta, 2r\sin\theta, -1) = \left(\frac{2r}{\sqrt{4r^2+1}}\cos\theta, \frac{2r}{\sqrt{4r^2+1}}\sin\theta, -\frac{1}{\sqrt{4r^2+1}}\right)$

$-1 \leq -\frac{1}{\sqrt{4r^2+1}} < 0$

Image of Gauss map : $S^2 \cap \{z<0\}$

If we choose $N = -\frac{\nabla F}{|\nabla F|}$, then image of Gauss map : $S^2 \cap \{z>0\}$

2b. Let $F(x,y,z) = x^2+y^2-z^2$, $S = F^{-1}(1)$

Choose $N = \frac{\nabla F}{|\nabla F|} = \frac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}}$

Take $x = \sqrt{1+z^2}\cos\theta$, $y = \sqrt{1+z^2}\sin\theta$

$N(z,\theta) = \left(\frac{\sqrt{1+z^2}}{\sqrt{1+2z^2}}\cos\theta, \frac{\sqrt{1+z^2}}{\sqrt{1+2z^2}}\sin\theta, -\frac{z}{\sqrt{1+2z^2}}\right)$

$-\frac{1}{\sqrt{2}} < -\frac{z}{\sqrt{1+2z^2}} < \frac{1}{\sqrt{2}}$

Image of Gauss map $= S^2 \cap \{-\frac{1}{\sqrt{2}} < z < \frac{1}{\sqrt{2}}\}$

If we choose $N = -\frac{\nabla F}{|\nabla F|}$, image of Gauss map has no change

2c. Let $F(x,y,z) = x^2+y^2-\cosh^2 z$, $S = F^{-1}(0)$

Choose $N = \frac{\nabla F}{|\nabla F|} = \frac{(x, y, -\cosh z\sinh z)}{\sqrt{x^2+y^2+\cosh^2 z\sinh^2 z}}$

Take $x = \cosh z\cos\theta$, $y = \cosh z\sin\theta$

$N(z,\theta) = \frac{1}{\sqrt{\cosh^2 z(1+\sinh^2 z)}}(\cosh z\cos\theta, \cosh z\sin\theta, -\cosh z\sinh z)$

$= \left(\frac{1}{\cosh z}\cos\theta, \frac{1}{\cosh z}\sin\theta, -\frac{\sinh z}{\cosh z}\right)$

$-1 < -\frac{\sinh z}{\cosh z} < 1$

Image of Gauss map $= S^2 \setminus \{(0,0,1), (0,0,-1)\}$

If we choose $N = -\frac{\nabla F}{|\nabla F|}$, image of Gauss map has no change.

3. Claim: p is umbilical point if and only if $4K_p = H_p^2$

Let k_1, k_2 be principle curvature at p.

$K_p = k_1 k_2$

$H_p = k_1 + k_2$

$(\Rightarrow)$ Suppose p is umbilical point (i.e. $k_1 = k_2$)

$K_p = k_1^2, \quad H_p = 2k_1$

$4K_p = H_p^2$

$(\Leftarrow)$ Suppose $4K_p = H_p^2$

$$4k_1k_2 = (k_1+k_2)^2$$
$$4k_1k_2 = k_1^2 + 2k_1k_2 + k_2^2$$
$$k_1^2 - 2k_1k_2 + k_2^2 = 0$$
$$(k_1-k_2)^2 = 0$$
$$k_1 = k_2$$